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SYSTEMS ANALYSIS

SERIES EXPANSION OF WEIGHTED PSEUDOINVERSE MATRICES AND ITERATIVE METHODS FOR CALCULATING WEIGHTED PSEUDOINVERSE MATRICES AND WEIGHTED NORMAL PSEUDOSOLUTIONS

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Weighted pseudoinverse matrices are expanded into matrix power series with negative exponents and arbitrary positive parameters. Based on this expansion, iterative methods for evaluating weighted pseudoinverse matrices and weighted normal pseudosolutions are designed and analyzed. The iterative methods for weighted normal pseudosolutions are extended to solving constrained least-squares problems.

Keywords: *weighted pseudoinverse matrices, weighted normal pseudosolutions, matrix power series, constrained least-squares problems, iterative methods.*

INTRODUCTION

The pseudoinverse Moore–Penrose matrix was expanded in [1] into a matrix power series with negative exponents. Expansions of weighted pseudoinverse matrices with positive definite and degenerate weights into matrix power series with negative exponents without the use of parameters for changing the convergence rate are obtained and analyzed in [2, 3], respectively. These results were used to generate iterative processes for weighted pseudoinverse matrices and weighted normal pseudosolutions with positive definite [2, 4] and degenerate [3, 5] weights. Here we propose and analyze the expansion of weighted pseudoinverse matrices with positive definite and degenerate weights into matrix power series with negative exponents and arbitrary positive parameters. This generalizes the previous results on weighted pseudoinverse matrices expanded into matrix power series with negative exponents. Based on the expansions obtained, we will generate and analyze iterative processes for weighted normal pseudosolutions and weighted pseudoinverse matrices with positive definite and degenerate weights that are a generalization of the iterative processes in the above-mentioned studies. We will show that iterative processes for weighted normal pseudosolutions can be used to solve a constrained least-squares problem.

The paper consists of five sections. In the first section, we present and analyze the properties of simmetrizable and weighted pseudoinverse matrices with positive definite and degenerate weights necessary for further presentation.

In the second section, we construct and analyze expansions of weighted pseudoinverse matrices with positive definite and degenerate weights into matrix power series. Also, we expand weighted pseudoinverse matrices into matrix power series, which include matrices inverse to symmetric and symmetrizable ones. The series are shown to converge with arbitrary positive parameters. The influence of the parameters on the convergence rate is analyzed.

In the third section, based on the expansions of weighted pseudoinverse matrices into matrix power series, we propose and analyze iterative processes for calculating weighted pseudoinverse matrices with positive definite and positive semidefinite weight matrices. Emphasis is on the convergence analysis of the iterative processes. The convergence rate is

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determined by estimating the error of the solution depending on the fixed number of iterations. It is shown that this estimate depends on a parameter.

In the fourth section, based on the expansions of weighted pseudoinverse matrices into matrix power series, we propose and analyze iterative processes for calculating weighted normal pseudosolutions with positive definite and degenerate weights. The estimates of decrease in the error of solution are obtained depending on the number of steps of the iterative process and on the iterative parameter.

In the fifth section, iterative processes for calculating weighted normal pseudosolutions are adapted to approximate the solution of a constrained least-squares problem and to find an *L*-pseudosolution (an *Lg*-pseudosolution, a coupled normal pseudosolution).

1. NOTATION, DEFINITIONS, WELL-KNOWN FACTS, AND AUXILIARY STATEMENTS

We assume that all scalars, vectors, matrices, and spaces to be used below are real. Let us introduce the notation and definitions necessary for further presentation. Let $\mathbb{R}^{m\times n}$ be a set of $m\times n$ real matrices.

Let us first define weighted pseudoinverse matrices with positive definite weights [6, 7]. Let $A\in\mathbb{R}^{m\times n}$, $X\in\mathbb{R}^{n\times m}$, $B\in\mathbb{R}^{m\times m}$, and $C\in\mathbb{R}^{n\times n}$ be symmetric positive definite matrices. Then a weighted pseudoinverse matrix for the matrix A is defined as a unique matrix $X=A^{+}_{BC}$ satisfying the following four conditions:

$$AXA=A, \tag{1}$$

$$XAX=X, \tag{2}$$

$$(BAX)^T=BAX, \tag{3}$$

$$(CXA)^T=CXA. \tag{4}$$

When $B=C=E$, where E is a unit matrix, the system of matrix equations (1)–(4) defines the Moore–Penrose pseudoinverse matrix [8, 9] for the matrix A; we denote it by A^{+}_{EE}.

Let us define weighted pseudoinverse matrices with degenerate weights. Let $A\in\mathbb{R}^{m\times n}$, $X\in\mathbb{R}^{n\times m}$, $B\in\mathbb{R}^{m\times m}$, and $C\in\mathbb{R}^{n\times n}$ be symmetric positive semidefinite matrices. Then the weighted pseudoinverse matrix for the matrix A in [10] is defined as the matrix $X=A^{+}_{BC}$ satisfying the following four conditions:

$$AXA=A, \tag{5}$$

$$XAX=X, \tag{6}$$

$$(BAX)^T=BAX, \tag{7}$$

$$(XAC)^T=XAC. \tag{8}$$

It is also established that the system of matrix equations (5)–(8) has a unique solution if and only if the relations

$$rk(BA)=rk(A),\quad rk(AC)=rk(A) \tag{9}$$

hold, where $rk(L)$ is the rank of the matrix L.

Let $\mathbb{R}^n$ be an *n*-dimensional vector space over a field of real numbers, where the vectors are $n\times 1$ matrices, H is a symmetric positive definite or positive semidefinite matrix, $\mathbb{R}^n(H)$ is a Euclidean space (in the case of a positive definite metric) or a pseudoeuclidean space (in the case of a non-negative metric) introduced by the scalar product $(u,v)_H=(Hu,v)_E$, where $(u,v)_E=u^T v$. We will introduce a norm (a semi-norm) in $\mathbb{R}^n(H)$ by the relation

$||u||_H = (u,u)_H^{1/2}$. In the case of a positive semidefinite matrix H, by $\overline{\mathbb{R}}^n(H) \subset \mathbb{R}^n(H)$ and $\overline{\mathbb{R}}^n(H_{EE}^+) \subset \mathbb{R}^n(H_{EE}^+)$ we will denote a subspace of vectors u satisfying the condition

$$H^{1/2} H_{EE}^{+1/2} u = u, \tag{10}$$

where $H_{EE}^{+1/2} = (H^{1/2})_{EE}^+$.

Hereafter, for positive semidefinite matrices H, we will use the notation $H_{EE}^{+p} = (H^p)_{EE}^+$, where p is an integer or fractional number.

The set of vectors satisfying condition (10) is not empty. Indeed, since $H^{1/2} H_{EE}^{+1/2}$ is a projectional matrix, the set (10) is an image of this matrix. Since null spaces of the matrices H, H_{EE}^+, and $H^{1/2} H_{EE}^{+1/2}$ coincide [11], the seminorms $||\cdot||_H$ and $||\cdot||_{H_{EE}^+}$ for vectors in $\mathbb{R}^n(H)$ and $\mathbb{R}^n(H_{EE}^+)$ become norms in $\overline{\mathbb{R}}^n(H)$ and $\overline{\mathbb{R}}^n(H_{EE}^+)$.

Let us define the norm of a rectangular matrix [5]. Let $A \neq 0 \in \mathbb{R}^{m \times n}$, H be a symmetric positive definite or positive semidefinite matrix of order m, V be a symmetric positive definite or positive semidefinite matrix of order n, and x be an arbitrary vector from $\mathbb{R}^n$. Let

$$rk(HA) = rk(A), \quad rk(AV) = rk(A). \tag{11}$$

If H and V are positive definite matrices, then conditions in (11) are certainly fulfilled.

For the set of matrices A satisfying (11), let us introduce the norm by the relation

$$||A||_{HV} = \sup_{x \neq 0} \frac{||H^{1/2} AVx||_{E_m}}{||x||_{E_n}}, \tag{12}$$

where $x \in \mathbb{R}^n$, and the subscript of a unit matrix denotes its dimension.

With such a definition, the norm of the matrix A is

$$||A||_{HV} = [\lambda_{\max}(VA^T HAV)]^{1/2}. \tag{13}$$

It is shown in [5] that the function $||\cdot||_{HV}$ defined by (12) is an additive matrix norm if conditions (11) hold. If conditions (11) (or one of them) are not fulfilled, then (12) defines the seminorm of the matrix A.

Let $A \in \mathbb{R}^{m \times p}$, $B \in \mathbb{R}^{p \times n}$, and $H \in \mathbb{R}^{m \times m}$, $V \in \mathbb{R}^{n \times n}$, and $M \in \mathbb{R}^{p \times p}$ be symmetric positive definite matrices. Then for the norm of a product of two rectangular matrices we have the estimate [2]

$$||AB||_{HV} \leq ||A||_{HM^{-1}} \, ||B||_{M^2 V} . \tag{14}$$

Now let us define the matrix norm of a square matrix [3]. Let $A \neq 0$ be an arbitrary square matrix of order n, and H be a symmetric positive semidefinite matrix of the same order, both satisfying the conditions

$$rk(HA) = rk(AH) = rk(A). \tag{15}$$

Let us define the norm of the matrix A as

$$||A||_H = \sup_{x \neq 0} \frac{||Ax||_H}{||x||_H} = \sup_{x \neq 0} \frac{||H^{1/2} A H_{EE}^{+1/2} H^{1/2} x||_E}{||H^{1/2} x||_E}, \tag{16}$$

where x is an arbitrary vector from $\overline{\mathbb{R}}^n(H)$.

With such a definition, the norm of the matrix A is

$$||A||_H = \left[\lambda_{\max}(H_{EE}^{+1/2} A^T H A H_{EE}^{+1/2})\right]^{1/2}. \tag{17}$$

Remark 1. If H is positive definite, it is necessary to replace the pseudoinverse matrix in (16), (17) with an inverse one.

Let A and B be square matrices of the same order, with the following condition being satisfied for the matrix B:

$$H_{EE}^{+1/2} H^{1/2} B = B, \tag{18}$$

where H is a symmetric positive semidefinite matrix of the same order as that of the matrices A and B. Then [3]

$$||AB||_H \le ||A||_H \; ||B||_H , \tag{19}$$

i.e., the function $||\cdot||_H$ defined by (16) is a multiplicative matrix norm if conditions (15) and (18) are fulfilled.

From (16) it follows that

$$||Ax||_H \le ||A||_H \; ||x||_H \;, \quad x \in \overline{\mathbb{R}}^n (H), \tag{20}$$

i.e., the matrix norm introduced by (16) is consistent with the vector norm.

Remark 2. If H is positive definite, then formulas (19) and (20) are true without additional conditions for the matrices A, H, and B, with $x \in \mathbb{R}^n (H)$.

Remark 3. From (13) and (17) it follows that the matrix norm (16) for the square matrices satisfying conditions (15) is a special case of the matrix norm (12) for rectangular matrices, which satisfies conditions (11) if we assume that A is a square matrix, $V = H_{EE}^{+1/2}$, and $x \in \overline{\mathbb{R}}^n (H)$. Therefore, we can use the notation $||A||_{HH_{EE}^{+1/2}}$ for the norm $||A||_H$ introduced by (16).

Let us define symmetrizable matrices with positive semidefinite symmetrizers [3].

Definition 1. A square matrix U is called left- or right-symmetrizable with symmetric positive semidefinite matrices M and N if the conditions

$$MU = U^T M, \quad rk\,(MU) = rk\,(U)\,; \tag{21}$$

$$UN = NU^T , \quad rk\,(UN) = rk\,(U)\,, \tag{22}$$

respectively, are satisfied.

Using (5) and (9), we can show that $rk\,(BAX) = rk\,(AX)$ and $rk\,(XAC) = rk\,(XA)$. Then condition (7) together with the first condition in (9) and condition (8) together with the second condition in (9) mean, respectively, that the matrix AX is left-symmetrizable with the symmetrizer B, and the matrix XA is right-symmetrizable with the symmetrizer C.

Remark 4. In the case of positive definite symmetrizers, the second conditions in (21) and (22) are certainly fulfilled. Moreover, it is easy to verify that if a matrix is left-symmetrizable with some symmetrizer, then it is right-symmetrizable with a matrix inverse to the symmetrizer matrix. Vice versa, if a matrix is right-symmetrizable with some symmetrizer, then it is left-symmetrizable with a matrix inverse to the symmetrizer matrix.

The authors have shown in [12] that the matrix U, left-symmetrizable with a positive definite symmetrizer H, can be diagonalized using the H-weighted orthogonal transformation, i.e., there is a matrix Q such that

$$Q^T HQ = E, \quad Q^T HUQ = \Lambda, \quad U = Q\Lambda Q^T H, \quad Q^T H = Q^{-1}, \tag{23}$$

where $\Lambda = \text{diag}\,(\lambda_i)$, λ_i are the eigenvalues of the matrix U, and

$$U^{-1} = Q\Lambda^{-1} Q^T H. \tag{24}$$

In a number of studies, symmetrizable matrices were defined and their properties were analyzed. For their brief characteristic see [13].

Remark 5. We will use Definition 1 for symmetrizable matrices. For some statements with degenerate symmetrizers, however, the second conditions (for matrix ranks) in (21) and (22) are not necessary. We will mention these cases below.

To substantiate the expansion of weighted pseudoinverse matrices with positive definite weights into matrix power series, the technique of weighted singular expansion of matrices designed in [14] is used. Namely, it is shown in [14] that two weighted orthogonal matrices $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ with weights B and C, respectively, such that

$$U^T BAV = D = \begin{cases} \| \operatorname{diag}(d_1, d_2, \ldots, d_r, 0, \ldots, 0) O_m^{n-m} \| & \text{if } m \le n, \\ \left\| \begin{matrix} \operatorname{diag}(d_1, d_2, \ldots, d_r, 0, \ldots, 0) \\ O_{m-n}^n \end{matrix} \right\| & \text{if } m \ge n, \end{cases} \tag{25}$$

$$A = UDV^T C, \tag{26}$$

exist for the matrix $A \in \mathbb{R}^{m \times n}$. Here $B \in \mathbb{R}^{m \times m}$ and $C \in \mathbb{R}^{n \times n}$ are arbitrary symmetric positive definite matrices, the columns of the matrices U and V are orthonormalized eigenvectors, in $\mathbb{R}^m(B)$ and $\mathbb{R}^n(C)$, of the symmetrizable matrices $AC^{-1}A^T B$ and $C^{-1}A^T BA$, respectively, and d_i are the square roots of the eigenvalues of the matrices $AC^{-1}A^T B$ if $m \le n$ and $C^{-1}A^T BA$ if $m \ge n$, $O_k^l \in \mathbb{R}^{k \times l}$ is zero matrix, and r is the rank of the matrix A.

It is also shown in [14] that the weighted pseudoinverse matrix for the matrix A is defined by the formula

$$A_{BC}^+ = VD_{EE}^+ U^T B, \tag{27}$$

where $D_{EE}^+ \in \mathbb{R}^{n \times m}$ is the pseudoinverse Moore–Penrose matrix for the matrix D, and D is defined by formula (25). Then D_{EE}^+ is a diagonal matrix, i.e., such that $d_{ij}^+ \ne 0 \Rightarrow i = j$, with nonzero elements $d_{ii}^+ = d_i^+$, $i = 1, \ldots, r$, where $d_i^+ = d_i^{-1}$, and d_i are defined in (25).

We will need estimates for the norm of the product of an arbitrary rectangular matrix and a left-symmetrizable matrix with a positive definite symmetrizer and for the norm of the product of an arbitrary rectangular matrix and a left-symmetrizable matrix with a positive definite symmetrizer, established in [2] and [12], respectively. Let us formulate the results obtained in [2, 12] as lemmas.

LEMMA 1. Let $A \in \mathbb{R}^{m \times n}$, $L \in \mathbb{R}^{m \times m}$ be a left-symmetrizable matrix with a positive definite symmetrizer $H \in \mathbb{R}^{m \times m}$, and $V \in \mathbb{R}^{n \times n}$ be any symmetric positive definite matrix. Then

$$\|LA\|_{HV} \le \|L\|_{HH^{-1/2}} \|A\|_{HV} = \rho(L) \|A\|_{HV}, \tag{28}$$

where $\rho(L)$ is the spectral radius of the matrix L.

LEMMA 2. Let $A \in \mathbb{R}^{m \times n}$, $L \in \mathbb{R}^{n \times n}$ be a left-symmetrizable matrix with a positive definite symmetrizer $V \in \mathbb{R}^{n \times n}$, and $H \in \mathbb{R}^{m \times m}$ be any symmetric positive definite matrix. Then

$$\|AL\|_{HV^{-1/2}} \le \|L\|_{VV^{-1/2}} \|A\|_{HV^{-1/2}} = \rho(L) \|A\|_{HV^{-1/2}}. \tag{29}$$

The papers [5, 15] analyze the properties of the matrix product of the right-symmetrizable matrix with a degenerate symmetrizer and an arbitrary rectangular matrix and the properties of the product matrix of an arbitrary rectangular matrix and a left-symmetrizable matrix with a degenerate symmetrizer. The results of obtained in [5, 15] are used to find the rate of convergence of iterative processes. Let us formulate them as lemmas.

LEMMA 3. Let a matrix $Y \in \mathbb{R}^{n \times m}$ satisfy the condition

$$C^{1/2} C_{EE}^{+1/2} Y = Y, \tag{30}$$

L be a square matrix of order n that satisfies the conditions

$$C^{1/2} C_{EE}^{+1/2} L = L, \tag{31}$$

$$LC = CL^T, \quad rk(LC) = rk(L), \tag{32}$$

LY be the matrix satisfying the first condition in (11) with $H = C_{EE}^+$, where C is a symmetric positive semidefinite matrix of order n that satisfies conditions (30)–(32), and V be a symmetric positive definite or positive semidefinite matrix of order m that satisfies the second condition in (11) for the matrix LY. Then the relation

$$||LY||_{C^+_{EE}V} \leq ||L||_{C^+_{EE}C^{1/2}} \, ||Y||_{C^+_{EE}V} = \rho(L)||Y||_{C^+_{EE}V} \tag{33}$$

holds for the matrix $LY \neq 0$.

LEMMA 4. Let a matrix $Y \in \mathbb{R}^{n \times m}$ satisfy the condition

$$YB^{1/2}B^{+1/2}_{EE} = Y, \tag{34}$$

L be a square matrix of order m satisfying the conditions

$$LB^{+1/2}_{EE}B^{1/2} = L, \tag{35}$$

$$BL = L^T B, \quad rk(BL) = rk(L), \tag{36}$$

YL be a matrix satisfying the second condition in (11) with $V = B^{+1/2}_{EE}$, where B is a symmetric positive semidefinite matrix of order m satisfying conditions (34)–(36), H be an arbitrary symmetric positive definite or positive semidefinite matrix of order n satisfying the first condition in (11) for the matrix YL, then the relations

$$||YL||_{HB^{+1/2}_{EE}} \leq ||Y||_{HB^{+1/2}_{EE}} \, ||L||_{BB^{+1/2}_{EE}} = \rho(L)||Y||_{HB^{+1/2}_{EE}} \tag{37}$$

hold for the matrix $YL \neq 0$.

To derive formulas for the expansion of weighted pseudoinverse matrices with positive definite weights into matrix power series, we will use the representation of weighted pseudoinverse matrices in terms of the coefficients of characteristic polynomials of symmetrizable matrices

$$A^+_{BC} = C^{-1}S_1 A^T B \tag{38}$$

obtained in [16], and the representation [17]

$$A^+_{BC} = CS_2 A^T B \tag{39}$$

for weighted pseudoinverse matrices with degenerate weights, where S_1 and S_2 are the polynomials of the matrices $C^{-1}A^T BA$ and $A^T BAC$, respectively, defined in [16] and [17].

It is shown in [16] that for positive definite B and C, the following equalities hold:

$$A^T = S_1 A^T BAC^{-1}A^T = A^T BAC^{-1}S_1 A^T, \quad A^+_{BC} AC^{-1}A^T B = C^{-1}A^T B,$$

$$A^T BAA^+_{BC} = A^T B. \tag{40}$$

For degenerate B and C, the following equalities [17] are true:

$$A^T = S_2 A^T BACA^T = A^T BACS_2 A^T,$$

$$A^+_{BC} ACA^T B = CA^T B, \quad A^T BAA^+_{BC} = A^T B. \tag{41}$$

We will need a statement about symmetrizability of matrices inverse to symmetrizable ones whose definition does not include a condition for the ranks of matrices.

LEMMA 5. Let the equalities

$$CA = A^T C, \quad BC = CB^T \tag{42}$$

hold for the matrices A and B, where A and B are nonsingular matrices and C is a symmetric positive definite or positive semidefinite matrix; then the same equalities are fulfilled for the matrices A^{-1} and B^{-1}, respectively.

To verify the statements of Lemma 5, it will suffice to multiply the first equality in (42) by A^{-1} on the right and by $(A^T)^{-1}$ on the left, and to multiply the second equality in (42) by B^{-1} on the left and by $(B^T)^{-1}$ on the right.

To prove the convergence of the iterative processes, we will use the properties of symmetrizable matrices established by the following lemmas.

LEMMA 6. Let A and B be the left(right)-symmetrizable matrices with the same degenerate symmetrizer. For the matrices AB or BA to be left(right)-symmetrizable with the same symmetrizer, it is necessary and sufficient that A and B be permutation matrices.

Proof. Let A and B be left-symmetrizable matrices with the same positive semidefinite symmetrizer matrix C, i.e.,

$$CA = A^T C, \quad rk\,(CA) = rk\,(A), \tag{43}$$

$$CB = B^T C, \quad rk\,(CB) = rk\,(B). \tag{44}$$

We seek for the necessary and sufficient conditions for the following equalities:

$$(CAB)^T = CAB, \quad rk\,(CAB) = rk\,(AB). \tag{45}$$

First we will prove the second equality in (45). Using the Frobenius inequality [18] and the fact that the rank of the product matrix does not exceed the rank of each of the multiplicand matrices, we obtain $rk\,(CA) + rk\,(AB) \le rk\,(A) + rk\,(CAB) \le rk\,(A) + rk\,(AB)$, whence by virtue of the second equality from (43) we have $rk\,(AB) \le rk\,(CAB) \le rk\,(AB)$, i.e., the second equality in (45).

Now let us establish the condition for the first equality in (45) to hold. By virtue of the first equalities in (43) and (44) we have

$$(CAB)^T = B^T A^T C = B^T CA = CBA.$$

Therefore, the first equality in (45) holds if and only if $AB = BA$.

In considering the matrix BA, it will suffice to use Eqs. (44) and the first equality in (43). The proof for the case of right-symmetrizable matrices is similar.

Lemma 6 is proved.

COROLLARY 1. From the proof of Lemma 6 it follows that if the left(right)-symmetrizable matrices A and B commute, then for the matrices AB and BA to be left(right)-symmetrizable with a symmetrizer C, it will suffice that only one of the second conditions in Eqs. (43) and (44) hold (any of the conditions for the ranks of the matrices in Eqs. (43) and (44)).

LEMMA 7. For a matrix L left-symmetrizable with a symmetric positive semidefinite matrix B, when the condition

$$LB_{EE}^{+1/2} B^{1/2} = L \tag{46}$$

is fulfilled, the following equality holds:

$$||L^n||_B \equiv ||L^n||_{BB_{EE}^{+1/2}} = ||L||_B^n \equiv ||L||_{BB_{EE}^{+1/2}}^n = [\rho\,(L)]^n, \quad n = 1, 2, \dots, \tag{47}$$

and for a matrix L right-symmetrizable with a symmetric positive semidefinite matrix C, for the condition

$$C^{1/2} C_{EE}^{+1/2} L = L \tag{48}$$

to be fulfilled, the following equality holds:

$$||L^n||_{C_{EE}^+} \equiv ||L^n||_{C_{EE}^+ C^{1/2}} = ||L||_{C_{EE}^+}^n \equiv ||L||_{C_{EE}^+ C^{1/2}}^n$$

$$= [\rho\,(L)]^n, \quad n = 1, 2, \dots, \tag{49}$$

where the matrix norms are defined by (16) and (17).

Proof. First, note that if the matrix L is left(right)-symmetrizable with some degenerate symmetrizer, then the matrix L^n $(n = 1, 2, \dots)$ is also left(right)-symmetrizable with the same symmetrizer and conditions (46) and (48) hold for the matrix L^n if they hold for the matrix L.

Indeed, if the equalities $BL = L^T B$ and $LC = CL^T$ hold, then we can establish by inspection that $BL^n = (L^n)^T B$ and $L^n C = C(L^n)^T$. To show that the equalities $rk(BL) = rk(L)$ and $rk(LC) = rk(L)$ yield $rk(BL^n) = rk(L^n)$ and $rk(L^n C) = rk(L^n)$, it will suffice to use the fact that the dimensions of null spaces of the matrices L and L^n coincide and $\dim(N(L)) = m - rk(L)$ [19], where $N(L)$ is a null space and m is the order of the matrix L. These statements yield $rk(L) = rk(L^n)$. Using the last equality and the Frobenius inequality [18], we obtain $rk(BL) + rk(L^n) \le rk(L) + rk(BL^n) \le rk(L^n) + rk(BL)$, whence, taking into account the equality $rk(BL) = rk(L)$, we have $rk(L^n) \le rk(BL^n) \le rk(L^n)$, i.e., $rk(BL^n) = rk(L^n)$. Similarly we obtain $rk(L^n C) = rk(L^n)$.

Let the matrix L be left-symmetrizable with the symmetrizer B and conditions (46) be satisfied for it. Then according to the aforesaid, the matrix L^n is also left-symmetrizable with the symmetrizer B and conditions (46) hold for it. In [15] it is shown that for the matrix L left-symmetrizable with the symmetric positive semidefinite matrix B, with condition (46) satisfied, the equality $||L||_{BB_{EE}^{+1/2}} \equiv ||L||_B = \rho(L)$ holds. Then $||L^n||_{BB_{EE}^{+1/2}} \equiv ||L^n||_B = \rho(L^n)$, $||L||^n_{BB_{EE}^{+1/2}} \equiv ||L||_B^n = [\rho(L)]^n$ and, taking into account (17) and the equality $\lambda_i(B_{EE}^{+1/2} L^T BLB_{EE}^{+1/2}) = \lambda_i^2(L)$ [14], where $\lambda_i(L)$ are the eigenvalues of the matrix L, we arrive at (47). Similarly, taking into account the equalities $||L||_{C_{EE}^{+}C^{1/2}} \equiv ||L||_C = \rho(L)$ and $\lambda_i(C^{1/2} L^T C_{EE}^{+} LC^{1/2}) = \lambda_i^2(L)$ obtained in [5] with (48), and formula (17), which has the following form for this case: $||A||_{C_{EE}^{+}} = [\lambda_{\max}(C^{1/2} A^T C_{EE}^{+} AC^{1/2})]^{1/2}$, we arrive at (49).

Lemma 7 is proved.

LEMMA 8. The ranks of the matrices AA_{BC}^{+} and $ACA^T B$ coincide.

The statement of the lemma can easily be verified by analogy with the proof of Lemma 4 from [15] as to the equality of the ranks of matrices coupled to weighted pseudoinversion with degenerate weights.

LEMMA 9. The matrices AA_{BC}^{+} and $ACA^T B$ commute, have a complete common system of eigenvectors and common null space with a maximum number of linearly independent vectors.

Proof. The matrix AA_{BC}^{+} can be represented as a polynomial [15]

$$AA_{BC}^{+} = f(ACA^T B)$$

$$= -\alpha_k^{-1}\left[(ACA^T B)^k + \alpha_1 (ACA^T B)^{k-1} + \ldots + \alpha_{k-1} ACA^T B\right], \tag{50}$$

whence the permutability of the matrices AA_{BC}^{+} and $ACA^T B$ follows.

Let us show that the matrices AA_{BC}^{+} and $ACA^T B$ have a complete common system of eigenvectors and that their null spaces coincide. Since according to (50) the matrix AA_{BC}^{+} can be represented as a polynomial of the matrix $ACA^T B$, all the eigenvectors of the matrix $ACA^T B$ are the eigenvectors of the matrix AA_{BC}^{+} [19]. Since the matrix AA_{BC}^{+} is idempotent, it is a matrix of simple structure [20] and, therefore, has m linearly independent eigenvectors. Then the matrices AA_{BC}^{+} and $ACA^T B$ have a common complete set of eigenvectors.

By virtue of the third equality in (41), $ACA^T B = ACA^T BAA_{BC}^{+}$, whence it follows that the eigenvectors corresponding to the zero eigenvalue of the matrix AA_{BC}^{+} are the eigenvectors corresponding to the zero eigenvalue of the matrix $ACA^T B$. And since the matrices AA_{BC}^{+} and $ACA^T B$ have a common system of linearly independent eigenvectors, the null spaces of these matrices coincide. Moreover, the idempotent matrix AA_{BC}^{+} has $m - rk(AA_{BC}^{+})$ linearly independent eigenvectors corresponding to the zero eigenvalue, i.e., the maximum number of linearly independent eigenvectors in its null space [19]. Lemma 8 states that the ranks of the matrices AA_{BC}^{+} and $ACA^T B$ are equal. Then these matrices have the maximum number of common linearly independent eigenvectors corresponding to the zero eigenvalue.

Lemma 9 is proved.

Let

$$Ax = f, \quad x \in \mathbb{R}^n, \quad f \in \mathbb{R}^m, \tag{51}$$

be a system of linear algebraic equations (SLAE) with an arbitrary matrix $A \in \mathbb{R}^{m \times n}$.

Definition 2. A vector x^+ that is a solution of the problem: find

$$\min_{x \in \Omega} ||x||_C, \quad \Omega = \operatorname{Arg} \min_{x \in \mathbb{R}^n} ||Ax - f||_B, \tag{52}$$

where B and C are symmetric positive definite matrices, is called a weighted normal pseudosolution with positive definite weights of system (51).

Remark 6. In [16] it is shown that problem (52) has a unique solution, which is determined by a weighted pseudoinverse matrix with positive definite weights expressed by conditions (1)–(4) and the right-hand side of system (51) according to the formula $x^+ = A_{BC}^+ f$.

Definition 3. A vector x^+ that is a solution of the problem: find

$$\min_{x \in \mathbb{R}^n (C_{EE}^+) \cap \Omega} ||x||_{C_{EE}^+}, \quad \Omega = \operatorname{Arg} \min_{x \in \mathbb{R}^n} ||Ax - f||_B, \tag{53}$$

where B and C_{EE}^+ are symmetric positive semidefinite matrices, is called a weighted normal pseudosolution with degenerate weights of system (51).

Remark 7. In [3] it is shown that problem (53) has a unique solution, which is determined by a weighted pseudoinverse matrix with degenerate weights defined by conditions (5)–(9) and the right-hand side of system (51) according to the formula $x^+ = A_{BC}^+ f$.

2. SERIES EXPANSION OF WEIGHTED PSEUDOINVERSE MATRICES

In the present section, we propose and analyze the expansion of weighted pseudoinverse matrices with positive definite and degenerate weights into matrix power series with negative exponents and arbitrary positive parameters.

2.1. Series Expansion of Weighted Pseudoinverse Matrices with Positive Definite Weights. Let us consider series expansion of the weighted pseudoinverse matrices defined by conditions (1)–(4). First the mathematical tools of the analysis are the weighted singular expansion of matrices defined in Sect. 1.

THEOREM 1. For an arbitrary matrix $A \neq 0 \in \mathbb{R}^{m \times n}$, symmetric positive definite matrices B and C such that $C^{-1} A^T BA$ is defined, and for a real number $0 < \alpha < \infty$, the following relations are true:

$$\alpha \sum_{k=1}^{\infty} (E + \alpha C^{-1} A^T BA)^{-k} C^{-1} A^T B = A_{BC}^+, \tag{54}$$

$$||A_{BC}^+ - A_{BC,j}^+||_{CB^{-1/2}} \leq \frac{(1 + \alpha d_*^2)^{-(j-1)}}{d_*}, \tag{55}$$

where $A_{BC,j}^+ = \alpha \sum_{k=1}^{j-1} (E + \alpha C^{-1} A^T BA)^{-k} C^{-1} A^T B$, d_* is the minimum nonzero diagonal element of the matrix D defined in (25).

Proof. Using (26) and the weighted orthogonality of the matrices U and V, we obtain

$$\alpha (E + \alpha C^{-1} A^T BA)^{-k} C^{-1} A^T B = \alpha (E + \alpha VD^T DV^T C)^{-k} VD^T U^T B.$$

Taking into account the representation of an inverse matrix for the product of two nonsingular matrices and the weighted orthogonality of the matrix V with the weight C, for any $k = 1, 2, \ldots$, based on the last equality we have

$$\alpha (E + \alpha C^{-1} A^T BA)^{-k} C^{-1} A^T B = \alpha V (E + \alpha D^T D)^{-k} D^T U^T B \tag{56}$$

and, therefore,

$$\alpha \sum_{k=1}^{\infty} (E+\alpha C^{-1} A^T BA)^{-k} C^{-1} A^T B = \alpha \sum_{k=1}^{\infty} V(E+\alpha D^T D)^{-k} D^T U^T B . \tag{57}$$

Since

$$(E+\alpha D^T D)^{-k} D^T = \mathrm{diag}\ \{(1+\alpha d_i^2)^{-k} d_i\} \tag{58}$$

and $(1+\alpha d_i^2)^{-1} < 1$ for $d_i > 0$, the series $\sum_{k=1}^{\infty} (E+\alpha D^T D)^{-k} D^T$ converges and

$$\sum_{k=1}^{\infty} (E+\alpha D^T D)^{-k} D^T = \alpha^{-1} D_{EE}^{+}, \tag{59}$$

where D_{EE}^{+} is the pseudoinverse Moore–Penrose matrix for the matrix D, which appears in the representation of the matrix A_{BC}^{+} according to (27).

By virtue of (27) and (59), from (57) we obtain

$$\alpha \sum_{k=1}^{\infty} (E+\alpha C^{-1} A^T BA)^{-k} C^{-1} A^T B = VD_{EE}^{+} U^T B = A_{BC}^{+},$$

i.e., expansion (54) of the weighted pseudoinverse matrix into a matrix power series.

Let us now prove estimate (55). Since

$$A_{BC}^{+} - A_{BC,j}^{+} = \alpha \sum_{k=j}^{\infty} (E+\alpha C^{-1} A^T BA)^{-k} C^{-1} A^T B,$$

with (56) we obtain

$$A_{BC}^{+} - A_{BC,j}^{+} = \alpha \sum_{k=j}^{\infty} V(E+\alpha D^T D)^{-k} D^T U^T B. \tag{60}$$

Put in (14) $M = E_n$, then by virtue of this relation, from (60) we have

$$\|A_{BC}^{+} - A_{BC,j}^{+}\|_{CB^{-1/2}} \le \alpha \|V\|_{CE_n} \left\| \sum_{k=j}^{\infty} (E_n + \alpha D^T D)^{-k} D^T U^T B \right\|_{E_n B^{-1/2}} .$$

Again we use (14), where suppose $M = E_m$ for the estimate of the second norm on the right-hand side of the last inequality. We obtain

$$\|A_{BC}^{+} - A_{BC,j}^{+}\|_{CB^{-1/2}}$$

$$\le \alpha \|V\|_{CE_n} \left\| \sum_{k=j}^{\infty} (E_n + \alpha D^T D)^{-k} D^T \right\|_{E_n E_m} \|U^T B\|_{E_m B^{-1/2}} . \tag{61}$$

Since U and V are weighted orthogonal matrices with the weights B and C, respectively, and the eigenvalues of the matrix product of two square matrices do not change after their permutation [18], by virtue of definition (13) of the matrix norm, from (61) we have

$$\|A_{BC}^{+} - A_{BC,j}^{+}\|_{CB^{-1/2}} \le \alpha \left\| \sum_{k=j}^{\infty} (E_n + \alpha D^T D)^{-k} D^T \right\|_{E_n E_m} . \tag{62}$$

Since the sum $\sum_{k=j}^{\infty}(1+\alpha d_i^2)^{-k} d_i = \frac{(1+\alpha d_i^2)^{-(j-1)}}{\alpha d_i}$ for $d_i \neq 0$ and is equal to zero for $d_i = 0$, taking into account (58), from inequality (62) we obtain estimate (55), which completes the proof of Theorem 1.

COROLLARY 2. Equality (54) yields the following relations:

$$A_{BC}^{+} = \alpha \sum_{k=1}^{\infty} C^{-1}(E+\alpha A^T BAC^{-1})^{-k} A^T B$$

$$= \alpha \sum_{k=1}^{\infty} C^{-1/2}(E+\alpha C^{-1/2} A^T BAC^{-1/2})^{-k} C^{-1/2} A^T B.$$

In [21], we estimated the sum of a finite number of terms of the matrix power series (54) for $\alpha \equiv 1$, i.e.,

$$\|A_{BC}^{+} - A_{BC,j}^{+}\|_{CB^{-1/2}} \leq \frac{(1+d_*^2)^{-(j-1)}}{d_*}. \tag{63}$$

Comparison of formulas (55) and (63) yields that the matrix power series (54) converges to a weighted pseudoinverse matrix faster for $\alpha > 1$ and more slowly for $\alpha < 1$ than the same series for $\alpha = 1$. These circumstances may be useful for the construction and realization of iterative processes based on series (54) for calculating weighted pseudoinverse matrices and weighted normal pseudosolutions.

Remark 8. Let in (55) $j = 2$. Then we can rearrange (55) as $\|A_{BC}^{+} - (\alpha^{-1}E + C^{-1}A^T BA)^{-1} C^{-1} A^T B\|_{CB^{-1/2}} \leq [d_*(1+\alpha d_*^2)]^{-1}$, whence $\lim_{\alpha \to \infty} (\alpha^{-1}E + C^{-1}A^T BA)^{-1} C^{-1} A^T B = A_{BC}^{+}$. Thus, from estimate (55) we get the limit representation of a weighted pseudoinverse matrix with positive definite weights, analyzed in [12].

Above, we have obtained the expansion of weighted pseudoinverse matrices with positive definite weights into matrix power series, where the mathematical tools are the weighted singular expansion of matrices [14]. To prove the statement below, we will use the representation of weighted pseudoinverse matrices with positive definite weights in terms of the coefficients of characteristic polynomials of symmetrizable matrices [16].

THEOREM 2. For an arbitrary matrix $A \in \mathbb{R}^{m \times n}$, symmetric positive definite matrices *B* and *C* such that $AC^{-1}A^T B$ is defined, and for a real number $0 < \alpha < \infty$, the limit $\lim_{i \to \infty} \alpha \sum_{k=1}^{i} C^{-1} A^T B (E + \alpha AC^{-1} A^T B)^{-k}$ exists and

$$\alpha \sum_{k=1}^{\infty} C^{-1} A^T B (E + \alpha AC^{-1} A^T B)^{-k} = A_{BC}^{+}. \tag{64}$$

Proof. Let $L = AC^{-1}A^T B$. The matrix L is left-symmetrizable with the symmetrizer *B*. Since the matrix L is a product of the positive semidefinite (positive definite in a special case) symmetric matrix $AC^{-1}A^T$ and the symmetric positive definite matrix *B*, its eigenvalues are non-negative [22]. Then the matrix $E + \alpha L$ for $\alpha > 0$ is nondegenerate and, therefore, $(E+\alpha L)^{-k}$ exists. Denote $\Lambda = \operatorname{diag}(\lambda_i)$, where λ_i are eigenvalues of the matrix L.

As stated above, in proving Theorem 2, we will use the representation of weighted pseudoinverse matrices with positive definite weights in terms of the coefficients of characteristic polynomials of symmetrizable matrices [16].

Using (23), (24), and the first equality in (40), we obtain

$$\alpha C^{-1} A^T B (E+\alpha L)^{-k} = \alpha C^{-1} S_1 A^T BAC^{-1} A^T B (E+\alpha L)^{-k}$$

$$= \alpha C^{-1} S_1^2 A^T BL^2 (E+\alpha L)^{-k} = \alpha C^{-1} S_1^2 A^T BQ\Lambda^2 Q^T BQ (E+\alpha\Lambda)^{-k} Q^T B \tag{65}$$

$$= \alpha C^{-1} S_1^2 A^T BQ\Lambda^2 (E+\alpha\Lambda)^{-k} Q^T B = \alpha^{-1} C^{-1} S_1^2 A^T BQ (\alpha\Lambda)^2 (E+\alpha\Lambda)^{-k} Q^T B.$$

Since $(\alpha\Lambda)^2(E+\alpha\Lambda)^{-k} = \mathrm{diag}\,[(\alpha\lambda_i)^2(1+\alpha\lambda_i)^{-k}]$ and $\alpha\lambda_i \geq 0$, the series $(\alpha\Lambda)^2 \sum\limits_{k=1}^{\infty}(E+\alpha\Lambda)^{-k}$ converges and

$$\sum_{k=1}^{\infty}(\alpha\Lambda)^2(E+\alpha L)^{-k} = \sum_{k=1}^{\infty}(E+\alpha\Lambda)^{-k}(\alpha\Lambda)^2 = \alpha\Lambda. \tag{66}$$

Based on (65), (66), (38), (23), and the permutability of the matrices S_1 and $A^T BAC^{-1}$, we have

$$\alpha\sum_{k=1}^{\infty}C^{-1}A^T B(E+\alpha L)^{-k} = C^{-1}S_1^2 A^T BQ\Lambda Q^T B = C^{-1}S_1^2 A^T BL$$

$$= C^{-1}S_1 A^T BAC^{-1}S_1 A^T B = C^{-1}S_1 A^T B = A_{BC}^+,$$

whence the statement of Theorem 2 follows.

COROLLARY 3. Equality (64) yields

$$A_{BC}^+ = \alpha\sum_{k=1}^{\infty}C^{-1}A^T(E+\alpha BAC^{-1}A^T)^{-k}B$$

$$= \alpha\sum_{k=1}^{\infty}C^{-1}A^T B^{1/2}(E+\alpha B^{1/2}AC^{-1}A^T B^{1/2})^{-k}B^{1/2}.$$

2.2. Series Expansion of Weighted Pseudoinverse Matrices with Degenerate Weights. Let us consider a series expansion of weighted pseudoinverse matrices defined by (5)–(9).

THEOREM 3. For an arbitrary matrix $A \in \mathbb{R}^{m\times n}$, symmetric positive semidefinite matrices $B \in \mathbb{R}^{m\times m}$ and $C \in \mathbb{R}^{n\times n}$ satisfying conditions (9), and for a real number $0<\alpha<\infty$, the following relations hold:

$$A_{BC}^+ = \alpha\sum_{k=1}^{\infty}C^{1/2}(E+\alpha C^{1/2}A^T BAC^{1/2})^{-k}C^{1/2}A^T B, \tag{67}$$

$$A_{BC}^+ = \alpha\sum_{k=1}^{\infty}(E+\alpha CA^T BA)^{-k}CA^T B, \tag{68}$$

$$A_{BC}^+ = \alpha\sum_{k=1}^{\infty}C(E+\alpha A^T BAC)^{-k}A^T B, \tag{69}$$

$$A_{BC}^+ = \alpha\sum_{k=1}^{\infty}CA^T B^{1/2}(E+\alpha B^{1/2}ACA^T B^{1/2})^{-k}B^{1/2}, \tag{70}$$

$$A_{BC}^+ = \alpha\sum_{k=1}^{\infty}CA^T B(E+\alpha ACA^T B)^{-k}, \tag{71}$$

$$A_{BC}^+ = \alpha\sum_{k=1}^{\infty}CA^T(E+\alpha BACA^T)^{-k}B. \tag{72}$$

Proof. First, let us prove relation (67). Denote $L = C^{1/2}A^T BAC^{1/2}$. The matrix L is symmetric and positive semidefinite; therefore, its eigenvalues are real and non-negative. Denote $\Lambda = \mathrm{diag}\,(\lambda_i)$, where λ_i are the eigenvalues of the matrix L. Since the matrix L is symmetric, an ordinary spectral decomposition holds for it, i.e., formulas (23) for $H \equiv E$. Let us consider one of the terms of series (67). With the first equality in (23) for $H \equiv E$ and the first equality in (41), we obtain

$$\alpha C^{1/2}(E+\alpha L)^{-k}C^{1/2}A^T B = \alpha C^{1/2}(E+\alpha L)^{-k}L^2 C^{1/2}S_2^2 A^T B$$

$$= \alpha C^{1/2}Q(E+\alpha\Lambda)^{-k}Q^T Q\Lambda^2 Q^T C^{1/2}S_2^2 A^T B = \alpha^{-1}C^{1/2}Q(E+\alpha\Lambda)^{-k}(\alpha\Lambda)^2 Q^T C^{1/2}S_2^2 A^T B. \tag{73}$$

By virtue of (73), (66), (23), (41), and (39) we have

$$\alpha \sum_{k=1}^{\infty} C^{1/2} (E+\alpha L)^{-k} C^{1/2} A^T B = C^{1/2} Q \Lambda Q^T C^{1/2} S_2^2 A^T B$$

$$= C^{1/2} L C^{1/2} S_2^2 A^T B = CA^T BACS_2^2 A^T B = CS_2 A^T B = A_{BC}^+,$$

i.e., we arrive at Eq. (67).

To prove Eq. (68), we use (67) and the Moore–Penrose property of pseudoinversion for a product of two matrices. It is well known (see, for example, [11]) that the equality

$$(MN)_{EE}^+ = N_{EE}^+ M_{EE}^+ \tag{74}$$

does not hold in the general case for a product of two arbitrary rectangular matrices. The necessary and sufficient conditions for this equality are given in the monograph [11]. For this equality to hold, the following relations should be fulfilled:

$$M_{EE}^+ MNN^T M^T = NN^T M^T, \quad NN_{EE}^+ M^T MN = M^T MN. \tag{75}$$

Let us consider the matrix $U = C^{1/2} (E+\alpha L)^{-1} C^{1/2} A^T B$, where, $L = C^{1/2} A^T BAC^{1/2}$ as above. Taking into account the equality $(A_{EE}^+)_{EE}^+ = A$, true for an arbitrary matrix A [11], we rearrange the matrix U as $U = (C_{EE}^{+1/2})_{EE}^+ (E+\alpha L)^{-1} C^{1/2} A^T B$. Suppose in (74) $N = C_{EE}^{+1/2}$ and $M = E + \alpha L$. It is easy to verify that such matrices M and N satisfy conditions (75). Then, taking into account the equality $C^{1/2} C_{EE}^{+1/2} = CC_{EE}^+$, we obtain $U = (C_{EE}^{+1/2} + \alpha C^{1/2} A^T BACC_{EE}^+)_{EE}^+ C^{1/2} A^T B$. Now we present the matrix U as $U = (C_{EE}^{+1/2} + \alpha C^{1/2} A^T BACC_{EE}^+)_{EE}^+ (C_{EE}^{+1/2})_{EE}^+ A^T B$. Suppose in (74) $N = C_{EE}^{+1/2} + \alpha C^{1/2} A^T BACC_{EE}^+$ and $M = C_{EE}^{+1/2}$. It is easy to verify that in this case the matrices M and N also satisfy conditions (75) and the matrix U can be rearranged as

$$U = (C_{EE}^+ + \alpha CC_{EE}^+ A^T BACC_{EE}^+)_{EE}^+ A^T B. \tag{76}$$

Taking into account (74), (75), and the equalities $(CC_{EE}^+)^2 = CC_{EE}^+$, $CC_{EE}^+ = C_{EE}^+ C$, and $C_{EE}^+ = C_{EE}^+ CC_{EE}^+$, we similarly obtain from (76)

$$U = CC_{EE}^+ (C_{EE}^+ + \alpha CC_{EE}^+ A^T BA)_{EE}^+ CC_{EE}^+ A^T B = CC_{EE}^+ (E + \alpha CA^T BA)_{EE}^+ CA^T B.$$

Since the matrix $CA^T BA$ is a product of two symmetric positive semidefinite matrices, its eigenvalues are non-negative and real [22]. Then the matrix $E + \alpha CA^T BA$ is nondegenerate; therefore, an inverse matrix exists for it and hence, by virtue of the latter equality, we can write $U = CC_{EE}^+ (E + \alpha CA^T BA)^{-1} CA^T B$. Since the matrices $(E+\alpha CA^T BA)^{-1}$ and $CA^T BA$ commute and $CC_{EE}^+ C = C$, taking into account the third equality in (41), we can represent the matrix U as

$$U = CC_{EE}^+ (E + \alpha CA^T BA)^{-1} CA^T BAA_{BC}^+$$

$$= CC_{EE}^+ CA^T BA (E + \alpha CA^T BA)^{-1} A_{BC}^+ = (E + \alpha CA^T BA)^{-1} CA^T B,$$

i.e.,

$$C^{1/2} (E+\alpha L)^{-1} C^{1/2} A^T B = (E + \alpha CA^T BA)^{-1} CA^T B, \tag{77}$$

where $L = C^{1/2} A^T BAC^{1/2}$ according to the notation above.

By virtue of (39) and the first equality from (41), we have the equality $C^{1/2} A^T B = LC_{EE}^{+1/2} A_{BC}^+$. Taking into account this equality, relation (77), and permutability of the matrices L and $(E+\alpha L)^{-1}$, we get

$$C^{1/2}(E+\alpha L)^{-k}C^{1/2}A^TB = C^{1/2}(E+\alpha L)^{-k}LC_{EE}^{+1/2}A_{BC}^{+}$$

$$= C^{1/2}(E+\alpha L)^{-1}L(E+\alpha L)^{-k+1}C_{EE}^{+1/2}A_{BC}^{+}$$

$$= (E+\alpha CA^TBA)^{-1}C^{1/2}L(E+\alpha L)^{-k+1}C_{EE}^{+1/2}A_{BC}^{+}$$

$$= (E+\alpha CA^TBA)^{-1}C^{1/2}(E+\alpha L)^{-1}L(E+\alpha L)^{-k+2}C_{EE}^{+1/2}A_{BC}^{+}$$

$$= (E+\alpha CA^TBA)^{-2}C^{1/2}(E+\alpha L)^{-1}L(E+\alpha L)^{-k+3}C_{EE}^{+1/2}A_{BC}^{+} =$$

$$\ldots = (E+\alpha CA^TBA)^{-k}CA^TB$$

for any $k = 2, 3, \ldots$, i.e.,

$$C^{1/2}(E+\alpha C^{1/2}A^TBAC^{1/2})^{-k}C^{1/2}A^TB$$

$$= (E+\alpha CA^TBA)^{-k}CA^TB, \quad k = 1, 2, \ldots. \tag{78}$$

Taking into account (78), we obtain (68) from (67).

Now let us prove (69). By virtue of (74), with (75) and Eqs. (39), (41), $(CC_{EE}^{+})^2 = CC_{EE}^{+}$, and $C_{EE}^{+}CC_{EE}^{+} = C_{EE}^{+}$, from (76) we obtain

$$U = C(CC_{EE}^{+} + \alpha CC_{EE}^{+}A^TBAC)_{EE}^{+}CC_{EE}^{+}A^TBACS_2A^TB$$

$$= C(E + \alpha CC_{EE}^{+}A^TBAC)_{EE}^{+}CC_{EE}^{+}A^TBACS_2A^TB.$$

Since the eigenvalues of the product matrix of two square matrices do not vary after their permutation and $CC_{EE}^{+}C = C$,

$$\lambda_i = (CC_{EE}^{+}A^TBAC) = \lambda_i(A^TBAC), \quad i = 1, \ldots, n.$$

The matrix A^TBAC is a product of two symmetric positive semidefinite matrices; therefore, its eigenvalues are real and non-negative [22]. Hence, the matrix $E + \alpha CC_{EE}^{+}A^TBAC$ is nondegenerate and has an inverse matrix, which commutes with the matrix $CC_{EE}^{+}A^TBAC$. Then, taking into account (74), (75), (39), and (41), we can rearrange the matrix U as

$$U = C(E + \alpha CC_{EE}^{+})^{-1}CC_{EE}^{+}A^TBACS_2A^TB$$

$$= CA^TBAC(E + \alpha CC_{EE}^{+}A^TBAC)^{-1}CC_{EE}^{+}S_2A^TB$$

$$= CA^TBAC(CC_{EE}^{+} + \alpha CC_{EE}^{+}A^TBAC)_{EE}^{+}S_2A^TB$$

$$= CA^TBAC(E + \alpha A^TBAC)^{-1}CC_{EE}^{+}S_2A^TB$$

$$= C(E + \alpha A^TBAC)^{-1}A^TBACS_2A^TB = C(E + \alpha A^TBAC)^{-1}A^TB,$$

i.e.,

$$C^{1/2}(E + \alpha C^{1/2}A^TBAC^{1/2})^{-1}C^{1/2}A^TB = C(E + \alpha A^TBAC)^{-1}A^TB\,. \tag{79}$$

With Eqs. (39), (41), (79), $C^{1/2}A^TB = LC_{EE}^{+1/2}A_{BC}^{+}$, and $C_{EE}^{+1/2}C = C^{1/2}$ and permutability of the matrices L and $(E+\alpha L)^{-k}$, A^TBAC and $(E + \alpha A^TBAC)^{-1}$, we have

$$C^{1/2}(E+\alpha L)^{-k}C^{1/2}A^TB = C^{1/2}(E+\alpha L)^{-k+1}L(E+\alpha L)^{-1}C_{EE}^{+1/2}A_{BC}^{+}$$

$$= C^{1/2}(E+\alpha L)^{-k+1}C^{1/2}C_{EE}^{+1/2}L(E+\alpha L)^{-1}C_{EE}^{+1/2}A_{BC}^{+}$$

$$= C^{1/2}(E+\alpha L)^{-k+1} C_{EE}^{+1/2} C (E+\alpha A^T BAC)^{-1} A^T B$$

$$= C^{1/2}(E+\alpha L)^{-k+1} C^{1/2} (E+\alpha A^T BAC)^{-1} A^T BACC_{EE}^{+} A_{BC}^{+}$$

$$= C^{1/2}(E+\alpha L)^{-k+1} LC^{1/2} (E+\alpha A^T BAC)^{-1} C_{EE}^{+} A_{BC}^{+}$$

$$= C^{1/2}(E+\alpha L)^{-k+2} L(E+\alpha L)^{-1} C^{1/2} (E+\alpha A^T BAC)^{-1} C_{EE}^{+} A_{BC}^{+}$$

$$= C^{1/2}(E+\alpha L)^{-k+2} C^{1/2} A^T BAC (E+\alpha A^T BAC)^{-2} C_{EE}^{+} A_{BC}^{+} =$$

$$\ldots = C(E+\alpha A^T BAC)^{-k} A^T B,$$

i.e.,

$$C^{1/2}(E+\alpha C^{1/2} A^T BAC^{1/2})^{-k} C^{1/2} A^T B$$

$$= C(E+\alpha A^T BAC)^{-k} A^T B, \quad k=1,2,\ldots. \tag{80}$$

Taking into account (80), we obtain (69) from (67).

The proof of the expansion of a weighted pseudoinverse matrix by formula (70) is similar to the proof of (67). To this end, an ordinary spectral resolution of the symmetric matrix $B^{1/2} ACA^T B^{1/2}$ is used for this. The expansions (71) and (72) are proved in the same fashion as (68) and (69), using formulas (74), (75), (39), and (41).

Theorem 3 is proved.

3. ITERATIVE METHODS FOR CALCULATING WEIGHTED PSEUDOINVERSE MATRICES

3.1. Iterative Methods for Calculating Weighted Pseudoinverse Matrices with Positive Definite Weights. To construct an iterative process, we will first use the expansion (54) of the weighted pseudoinverse matrix defined by conditions (1) – (4) into a matrix power series. Based on this series, we obtain the following iterative process:

$$X_0 = 0; \quad X_{k+1} = \Psi^{-p} X_k + \alpha \sum_{i=1}^{p} \Psi^{-i} C^{-1} A^T B,$$

$$\Psi = E + \alpha C^{-1} A^T BA, \quad k = 0,1,\ldots. \tag{81}$$

THEOREM 4. The iterative process (81) for $0 < \alpha < \infty$ converges, and the estimate

$$\| A_{BC}^{+} - X_{k+1} \|_{CV} \le q^{p(k+1)} \| A_{BC}^{+} \|_{CV} \tag{82}$$

holds, where

$$q = \rho\left[A_{BC}^{+} A (E+\alpha C^{-1} A^T BA)^{-1} \right] = \left[1 + \alpha \lambda_{\min}^{*} (C^{-1} A^T BA) \right]^{-1} < 1, \tag{83}$$

$\lambda_{\min}^{*}(L)$ is the minimum nonzero eigenvalue of the matrix L, the matrix C appears in the definition of the weighted pseudoinverse matrix according to condition (4), and $V \in \mathbb{R}^{m \times m}$ is an arbitrary symmetric positive definite matrix.

Proof. The convergence of the sequence of matrices defined by (81) to the weighted pseudoinverse matrix A_{BC}^{+} defined by the conditions (1)–(4), as $k \to \infty$, follows from the fact that this sequence is based on the matrix series (54). Let us prove the validity of the estimate (82).

First let us prove the equality

$$A_{BC}^{+} - X_{k+1} = \Psi^{-p} (A_{BC}^{+} - X_k), \quad \Psi = E + \alpha C^{-1} A^T BA, \quad k = 0,1,\ldots \tag{84}$$

by induction. Let $p=1$. Then, with the third equality in (40), from (81) we have

$$A^+_{BC} - X_{k+1} = A^+_{BC} - \Psi^{-1} X_k - \alpha \Psi^{-1} C^{-1} A^T B$$

$$= \Psi^{-1} (\Psi A^+_{BC} - X_k - \alpha C^{-1} A^T B)$$

$$= \Psi^{-1} (A^+_{BC} - X_k - \alpha C^{-1} A^T B + \alpha C^{-1} A^T B A A^+_{BC}) = \Psi^{-1} (A^+_{BC} - X_k).$$

Let Eq. (84) hold for $p=n$. Let us show that this equality is true for $p=n+1$. With the third equality in (40), from (81) we obtain

$$A^+_{BC} - X_{k+1} = A^+_{BC} - \Psi^{-(n+1)} X_k - \alpha \sum_{i=1}^{n+1} \Psi^{-i} C^{-1} A^T B$$

$$= \Psi^{-n} (A^+_{BC} - X_k) - \Psi^{-(n+1)} X_k + \Psi^{-n} X_k - \alpha \Psi^{-(n+1)} C^{-1} A^T B$$

$$= \Psi^{-n} (A^+_{BC} - X_k) + \Psi^{-(n+1)} (-X_k - \alpha C^{-1} A^T B + \Psi X_k)$$

$$= \Psi^{-n} (A^+_{BC} - X_k) + \Psi^{-(n+1)} (-X_k + X_k - \alpha C^{-1} A^T B + \alpha C^{-1} A^T B A X_k)$$

$$= \Psi^{-(n+1)} (A^+_{BC} - X_k + \alpha C^{-1} A^T B A A^+_{BC} - \alpha C^{-1} A^T B A X_k - \alpha C^{-1} A^T B$$

$$+ \alpha C^{-1} A^T B A X_k) = \Psi^{-(n+1)} (A^+_{BC} - X_k).$$

In [2] it is shown that the matrix $A^+_{BC} A$ can be presented as

$$A^+_{BC} A = -\alpha_k^{-1} \left[(C^{-1} A^T B A)^k + \alpha_1 (C^{-1} A^T B A)^{k-1} + \ldots + \alpha_{k-1} C^{-1} A^T B A \right],$$

whence it is clear that the matrices $A^+_{BC} A$ and $C^{-1} A^T B A$ commute. Then the matrices $A^+_{BC} A$ and Ψ^{-1} commute too. Taking into account Eqs. (40), the permutability of the matrices $A^+_{BC} A$ and Ψ^{-1}, and that $A^+_{BC} A$ is an idempotent matrix, it is easy to verify that $A^+_{BC} A X_k = X_k$, $k = 0, 1, \ldots$, whence we can rearrange (84) as

$$A^+_{BC} - X_{k+1} = (A^+_{BC} A \Psi^{-1})^p (A^+_{BC} - X_k), \quad k = 0, 1, \ldots. \tag{85}$$

In [2] it is shown that the matrix $A^+_{BC} A \Psi^{-1}$ is left-symmetrizable with the symmetrizer C. Then the matrix $(A^+_{BC} A \Psi^{-1})^p$ is also left-symmetrizable with the symmetrizer C, whence, based on Lemma 1, from (85) we obtain

$$\| A^+_{BC} - X_{k+1} \|_{CV} \le \| (A^+_{BC} A \Psi^{-1})^p \|_{CC^{-1/2}} \| A^+_{BC} - X_k \|_{CV} \le$$

$$\ldots \le \| (A^+_{BC} A \Psi^{-1})^p \|^{k+1}_{CC^{-1/2}} \| A^+_{BC} - X_0 \|_{CV}. \tag{86}$$

In [12] it is shown that $\| U^n \|_{CC^{-1/2}} = \| U \|^n_{CC^{-1/2}}$ for symmetrizable matrices, whence based on Lemma 1 and the initial approximation of the iterative process (81) we obtain the estimate (82).

In [2] it is shown that the absolute values of all the eigenvalues of the matrix $A^+_{BC} A (E + C^{-1} A^T B A)^{-1}$ are less than unity. This can be proved similarly for the matrix $L = A^+_{BC} A (E + \alpha C^{-1} A^T B A)^{-1}$ for $\alpha > 0$. To this end, it will suffice to use the weighted spectral resolution (23) of the symmetrizable matrices $A^+_{BC} A$ and $(E + \alpha C^{-1} A^T B A)^{-1}$ and that the matrices $A^+_{BC} A$ and $C^{-1} A^T B A$ have identical ranks, the common system of eigenvectors, and their null spaces coincide [2]. As a result, we establish that all the eigenvalues of the matrix L are nonnegative, less than unity, and the maximum eigenvalue of this matrix is $\lambda_{\max}(L) = [1 + \alpha \lambda^*_{\min} (C^{-1} A^T B A)]^{-1}$, whence (83) follows.

Theorem 4 is proved.

From (83) it follows that the value of q depends on the parameter α and decreases with increase in this parameter. To accelerate the convergence of the iterative process, it is necessary to take a sufficiently large α. But an increase in the parameter α increases the conditioning of the matrix $E + \alpha C^{-1} A^T BA$, which is telated to the accuracy of calculating the matrix inverse to the matrix $E + \alpha C^{-1} A^T BA$. Therefore, the choice of the parameter α is of great importance for the design and generation of iterative processes.

To design the iterative process, we will now use the expansion (64) of the weighted pseudoinverse matrix defined by conditions (1)–(4) into a matrix power series. Based on this series, we obtain the following iterative process:

$$X_0 = 0; \quad X_{k+1} = X_k \Psi^{-p} + \alpha C^{-1} A^T B \sum_{i=1}^{p} \Psi^{-i},$$

$$\Psi = E + \alpha A C^{-1} A^T B, \quad k = 0, 1, \ldots . \tag{87}$$

THEOREM 5. The iterative process (87) converges for $0 < \alpha < \infty$, and the following estimate holds:

$$\| A_{BC}^{+} - X_{k+1} \|_{HB^{-1/2}} \le q^{p(k+1)} \| A_{BC}^{+} \|_{HB^{-1/2}}, \tag{88}$$

where

$$q = \rho \left[A A_{BC}^{+} (E + \alpha A C^{-1} A^T B)^{-1} \right] = \left[1 + \alpha \lambda_{\min}^{*} (A C^{-1} A^T B) \right]^{-1} < 1, \tag{89}$$

$H \in \mathbf{R}^{n \times n}$ is an arbitrary symmetric positive definite matrix, and the matrix $B \in \mathbb{R}^{m \times m}$ appears in the definition of a weighted pseudoinverse matrix according to condition (3).

Proof. The convergence of the sequence of matrices defined by (87) to the weighted pseudoinverse matrix A_{BC}^{+} defined by conditions (1)–(4) as $k \to \infty$ follows from the fact that this sequence is based on the matrix series (64). Let us prove the estimate (88).

First let us prove the equality

$$A_{BC}^{+} - X_{k+1} = (A_{BC}^{+} - X_k) \Psi^{-p}, \quad \Psi = E + \alpha A C^{-1} A^T B, \quad k = 0, 1, \ldots . \tag{90}$$

by induction. Let $p = 1$. Then, taking into account the second equality in (40), from (87) we have

$$A_{BC}^{+} - X_{k+1} = A_{BC}^{+} - X_k \Psi^{-1} - \alpha C^{-1} A^T B \Psi^{-1}$$

$$= (A_{BC}^{+} \Psi - X_k - \alpha C^{-1} A^T B) \Psi^{-1}$$

$$= (A_{BC}^{+} - X_k - \alpha C^{-1} A^T B + \alpha A_{BC}^{+} A C^{-1} A^T B) \Psi^{-1} = (A_{BC}^{+} - X_k) \Psi^{-1}.$$

Let equality (90) hold for $p = n$. Let us show that it is also true for $p = n + 1$. Taking into account the second equality in (40), from (87) we obtain

$$A_{BC}^{+} - X_{k+1} = A_{BC}^{+} - X_k \Psi^{-(n+1)} - \alpha C^{-1} A^T B \sum_{i=1}^{n+1} \Psi^{-i}$$

$$= (A_{BC}^{+} - X_k) \Psi^{-n} - X_k \Psi^{-(n+1)} + X_k \Psi^{-n} - \alpha C^{-1} A^T B \Psi^{-(n+1)}$$

$$= (A_{BC}^{+} - X_k) \Psi^{-n} + (-X_k - \alpha C^{-1} A^T B + X_k \Psi) \Psi^{-(n+1)}$$

$$= (A_{BC}^{+} - X_k) \Psi^{-n} + (-X_k + X_k - \alpha C^{-1} A^T B + \alpha X_k A C^{-1} A^T B) \Psi^{-(n+1)}$$

$$= (A_{BC}^{+} - X_k + \alpha A_{BC}^{+} A C^{-1} A^T B - \alpha X_k A C^{-1} A^T B - \alpha C^{-1} A^T B$$

$$+ \alpha X_k A C^{-1} A^T B) \Psi^{-(n+1)} = (A_{BC}^{+} - X_k) \Psi^{-(n+1)}.$$

In [12] it is shown that the matrices AA^+_{BC} and $AC^{-1}A^TB$ commute, have a complete common system of eigenvectors, and their null spaces coincide. Then the matrices AA^+_{BC} and $(E+\alpha AC^{-1}A^TB)^{-1}$ commute too. Taking into account the third equality in (40) and the permutability of the matrices AA^+_{BC} and $(E+\alpha AC^{-1}A^TB)^{-1}$, it is easy to verify that the equalities $X_k = X_k AA^+_{BC}$, $k=0,1,\dots$, hold for the matrices X_k defined by (87). Using these equalities, the permutability of the matrices AA^+_{BC} and $(E+\alpha AC^{-1}A^TB)^{-1}$ symmetrizable with the symmetrizer B, Eq. (2), and that the matrix AA^+_{BC} is idempotent, from (90) we obtain

$$A^+_{BC} - X_{k+1} = (A^+_{BC} - X_k)(AA^+_{BC}\Psi^{-1})^p,$$

$$\Psi = E + \alpha AC^{-1}A^TB, \quad k=0,1,\dots. \tag{91}$$

Now let us show that all the eigenvalues of the matrix $L = AA^+_{BC}\Psi^{-1}$ are less than unity. To this end, we use the weighted spectral resolution of the symmetrizable matrices (23). Since the matrices AA^+_{BC} and Ψ^{-1} are symmetrizable with the symmetrizer B, have a complete common system of eigenvectors, and their null spaces coincide, according to (23) we have

$$L = QJQ^TBQ(E+\alpha\Lambda)^{-1}Q^TB = QJ(E+\alpha\Lambda)^{-1}Q^TB = QDQ^TB, \tag{92}$$

where $J = \mathrm{diag}\,\{1,\dots,1,0,\dots,0\}$ are the eigenvalues of the idempotent matrix AA^+_{BC} with the number of zero eigenvalues $m-r$, r is the rank of the matrices AA^+_{BC} and $AC^{-1}A^TB$, $\Lambda = \mathrm{diag}\,\{\lambda_i > 0\,(i=1,2,\dots,r), 0,\dots,0\}$ are the eigenvalues of the matrix $AC^{-1}A^TB$, and $D = \mathrm{diag}\,\{(1+\alpha\lambda_i)^{-1}\ (i=1,2,\dots,r),\ 0,\dots,0\}$. Thus, all the eigenvalues of the matrix L are nonnegative and the maximum eigenvalue of this matrix is

$$\lambda_{\max}(L) = \left[1+\alpha\lambda^*_{\min}(AC^{-1}A^TB)\right]^{-1} < 1, \tag{93}$$

where $\lambda^*_{\min}(K)$ is the least nonzero eigenvalue of the matrix K.

Since the matrix L is symmetrizable with the symmetrizer B, taking into account Lemma 2, the equality $||L^n||_{VV^{-1/2}} = ||L||^n_{VV^{-1/2}}$, $n=1,2,\dots$ [13], relation (93), and zero initial approximation of the iterative process (87), from (91) we obtain (88), as was to be shown.

Remark 9. In Theorems 4 and 5, q are determined by (83) and (89), respectively. Since the non-negative eigenvalues of the product matrix do not change after the permutation of multiplicand matrices [18], the values of q defined in Theorems 4 and 5 coincide. In (82) and (88), the matrices V and H, respectively, are arbitrary symmetric positive definite. Then if $V = B^{-1/2}$ in (82) and $H = C$ in (88), then formulas (82) and (88) are identical.

3.2. Iterative Methods for Calculating Weighted Pseudoinverse Matrices with Degenerate Weights. To generate an iterative process, we will first use expansion (68) of the weighted pseudoinverse matrix defined by conditions (5)–(9) into a matrix power series. Based on this series, we obtain the following iterative process:

$$X_0 = 0; \quad X_{k+1} = \Psi^{-p}X_k + \alpha\sum_{i=1}^{p}\Psi^{-i}CA^TB,$$

$$\Psi = E + \alpha CA^TBA, \quad k=0,1,\dots. \tag{94}$$

THEOREM 6. The iterative process (94) converges for $0<\alpha<\infty$, and the following estimate holds:

$$||A^+_{BC} - X_{k+1}||_{C^+_{EE}V} \le q^{p(k+1)}||A^+_{BC}||_{C^+_{EE}V}\ , \tag{95}$$

where $q=\rho[A_{BC}^{+}A(E+\alpha CA^{T}BA)^{-1}]=[1+\alpha\lambda_{\min}^{*}(CA^{T}BA)]^{-1}<1$, $\lambda_{\min}^{*}(L)$ is the minimum nonzero eigenvalue of the matrix L, the matrix C appears in the definition of a weighted pseudoinverse matrix according to condition (8) and the second condition in (9), and $V\in\mathbb{R}^{m\times m}$ is an arbitrary symmetric positive definite or positive semidefinite matrix satisfying the second condition in (11) for the matrix $A_{BC}^{+}-X_{k+1}$.

Proof. Convergence of the sequence of matrices defined by (94) to the weighted pseudoinverse matrix A_{BC}^{+} defined by conditions (5)–(9) as $k\to\infty$ follows from the fact that this sequence is based on the matrix series (68). Let us prove the validity of the estimate (95).

With the second equality from (41), the equality

$$A_{BC}^{+}-X_{k+1}=\Psi^{-p}(A_{BC}^{+}-X_{k}),\quad \Psi=E+\alpha CA^{T}BA,\quad k=0,1,\dots, \tag{96}$$

can be proved similarly to (84).

In [5] it is shown that the matrices $A_{BC}^{+}A$ and $E+CA^{T}BA$ are permutable. Then the matrices $A_{BC}^{+}A$ and Ψ^{-1} commute too. Taking into account this circumstance and the second equality in (41), it is easy to verify that

$$A_{BC}^{+}AX_{k}=X_{k},\quad k=0,1,\dots. \tag{97}$$

By virtue of equalities (97) and (6), the permutability of the matrices $A_{BC}^{+}A$ and Ψ^{-1}, and the fact that the matrix $A_{BC}^{+}A$ is idempotent, from (96) we obtain

$$A_{BC}^{+}-X_{k+1}=(A_{BC}^{+}A\Psi^{-1})^{p}(A_{BC}^{+}-X_{k}),\ \Psi=E+\alpha CA^{T}BA,\ k=0,1,\dots. \tag{98}$$

In [3] it is shown that eigenvalues of the matrix $A_{BC}^{+}A(E+CA^{T}BA)^{-1}$ are nonnegative, less than unity, and the spectral radius of this matrix is defined. It can similarly be proved that the eigenvalues of the matrix $A_{BC}^{+}A\Psi^{-1}$ for $\alpha>0$ are nonnegative, less than unity, and

$$q=\rho(A_{BC}^{+}A\Psi^{-1})=\left[1+\alpha\lambda_{\min}^{*}(CA^{T}BA)\right]^{-1}<1. \tag{99}$$

We will estimate the approximation error $Z=A_{BC}^{+}-X_{k+1}$ of the $(k+1)$th iteration to the weighted pseudoinverse matrix in norm $\|\cdot\|_{C_{EE}^{+}V}$ defined by (12) and (13). Let us use Lemma 3, where we put $L=A_{BC}^{+}A\Psi^{-p}$ and $Y=A_{BC}^{+}-X_{k}$. We check up the conditions of Lemma 3 imposed on the matrices Y, L, and $Z=LY$, for which relation (33) is fulfilled. For the matrix Y, condition (30) is fulfilled due to formula (39) for the matrix A_{BC}^{+}, Eqs. (97), and the elementary equality $C^{1/2}C_{EE}^{+1/2}C=C$. For the matrix L, condition (31) holds by virtue of (39) and the equality $C^{1/2}C_{EE}^{+1/2}C=C$. Since the matrix $E+\alpha CA^{T}BA$ is right-symmetrizable with the symmetrizer C, by virtue of Lemma 5 the matrix Ψ^{-p} is also right-symmetrizable with the symmetrizer C. The matrix $A_{BC}^{+}A$ is right-symmetrizable with the symmetrizer C by virtue of (8). Moreover, based on Lemma 2 from [3], the matrices $A_{BC}^{+}A$ and Ψ^{-1} commute. Then the first condition in (32) is satisfied for the matrix L by virtue of Lemma 6. To show that the second condition in (32) holds for the matrix L, let us rearrange it as $L=\Psi^{-p}A_{BC}^{+}A$, then we use the Frobenius inequality [18], whence $rk(\Psi^{-p}A_{BC}^{+}A)+rk(AC)\le rk(A)+rk(\Psi^{-p}A_{BC}^{+}AC)$. From this relation, the second equality in (9), and the estimate for the ranks of the product matrix in terms of the ranks of multiplicand matrices, the second condition in (32) for the matrix L follows.

It remains to show that the first axiom of the matrix norm $\|\cdot\|_{C_{EE}^{+}V}$ is fulfilled for the matrix Z. Since we assumed that the matrix V satisfies the second condition in (11) for the matrix Z, for the inequality $\|\cdot\|_{C_{EE}^{+}V}>0$ to hold it will suffice to show that the first condition in (11) is satisfied for the matrices Z and C_{EE}^{+}, i.e.,

$$rk(C_{EE}^{+}Z)=rk(Z), \tag{100}$$

where C^+_{EE} is the Moore–Penrose pseudoinverse matrix for the matrix C, which appears in the definition of a weighted pseudoinverse matrix according to conditions (8) and (9). With (39) and the equality $CC^+_{EE}C = C$, we have $rk(Z) = rk(CC^+_{EE}Z) \le rk(C^+_{EE}Z) \le rk(Z)$, whence (100) follows.

Therefore, all the conditions of Lemma 3 for the matrices Y, L, and LY are satisfied. Then relation (33) holds for the matrix LY, whence we obtain from (98)

$$||A^+_{BC} - X_{k+1}||_{C^+_{EE}V} \le ||(A^+_{BC}A\Psi^{-1})^p||_{C^+_{EE}C^{1/2}} \, ||A^+_{BC} - X_k||_{C^+_{EE}V}$$

$$= \rho(A^+_{BC}A\Psi^{-1})^p \, ||A^+_{BC} - X_k||_{C^+_{EE}V} . \tag{101}$$

The matrix $A^+_{BC}A$ is right-symmetrizable with the symmetrizer C by virtue of condition (8). Based on Lemma 5, the second equality in (42) holds for the matrix Ψ^{-1}. As indicated above, the matrices $A^+_{BC}A$ and Ψ^{-1} commute and the second condition in (43) holds for the matrix $A^+_{BC}A$. Then according to Corollary 1 of Lemma 6, the matrix $A^+_{BC}A\Psi^{-1}$ is right-symmetrizable with the simmmetrizer C, and condition (48) is satisfied for it by virtue of (39). Then based on Lemma 7 (formula (49)), with (99), from (101) we obtain (95), which completes the proof of Theorem 6.

To generate the iterative process, we will use the matrix power series expansion (71) of the weighted pseudoinverse matrix defined by conditions (5)–(9). Based on this series, we obtain the following iterative process:

$$X_0 = 0; \quad X_{k+1} = X_k\Psi^{-p} + \alpha CA^T B \sum_{i=1}^{p} \Psi^{-i},$$

$$\Psi = E + \alpha ACA^T B, \quad k = 0, 1, \ldots . \tag{102}$$

THEOREM 7. The iterative process (102) converges for $0 < \alpha < \infty$, and the following estimate holds:

$$||A^+_{BC} - X_{k+1}||_{HB^{+1/2}_{EE}} \le q^{p(k+1)} ||A^+_{BC}||_{HB^{+1/2}_{EE}}, \tag{103}$$

where $q = \rho(AA^+_{BC}\Psi^{-1}) = [1 + \alpha\lambda^*_{\min}(ACA^T B)]^{-1} < 1$, $\lambda^*_{\min}(L)$ is the minimum nonzero eigenvalue of the matrix L, $H \in \mathbb{R}^{n\times n}$ is an arbitrary symmetric positive definite or positive semidefinite matrix satisfying the first condition in (11) for the matrix $A^+_{BC} - X_{k+1}$, the matrix B appears in the definition of the weighted pseudoinverse matrix according to condition (7) and to the first condition in (9).

Proof of this theorem follows from the same scheme as the proof of Theorem 6. First we prove the equality

$$A^+_{BC} - X_{k+1} = (A^+_{BC} - X_k)\Psi^{-p}, \quad \Psi = E + \alpha ACA^T B, \quad k = 0, 1, \ldots, \tag{104}$$

by induction, for which we use the third equality in (41).

Then based on Lemma 9 and the third equality in (41) we show that the equalities $X_k = X_k AA^+_{BC}$, $k = 0, 1, \ldots$, hold for X_k defined by (102). By virtue of these equalities, Eq. (6), the permutability of the matrices AA^+_{BC} and Ψ^{-1}, and the fact that the matrix AA^+_{BC} is idempotent, from (104) we obtain

$$A^+_{BC} - X_{k+1} = (A^+_{BC} - X_k)(AA^+_{BC}\Psi^{-1})^p, \quad \Psi = E + \alpha ACA^T B, \quad k = 0, 1, \ldots . \tag{105}$$

By analogy with how this was done for the matrix $A^+_{BC}A(E + CA^T BA)$ in [3], it can be shown that the eigenvalues of the matrix $AA^+_{BC}\Psi^{-1}$ for $\alpha > 0$ are nonnegative, less than unity, and

$$q = \rho(AA^+_{BC}\Psi^{-1}) = \left[1 + \alpha\lambda^*_{\min}(ACA^T B)\right]^{-1} < 1. \tag{106}$$

To obtain estimate (103) from (105), we will use Lemma 4 after checking the conditions under which the statement of this lemma holds. Then based on (37) we obtain

$$||A_{BC}^{+} - X_{k+1}||_{HB_{EE}^{+1/2}} \le \rho(AA_{BC}^{+}\Psi^{-1})^{p} ||A_{BC}^{+} - X_{k}||_{HB_{EE}^{+1/2}}. \tag{107}$$

Finally, in view of Lemmas 5 and 7 and formula (106), we obtain estimate (103) from (107), which completes the proof of Theorem 7.

Remark 10. In Theorems of 6 and 7, q are determined by formulas (99) and (106), respectively. Since non-negative eigenvalues of the product matrix do not change after permutation of the multiplicand matrices [18], the values of q defined in Theorems 6 and 7 coincide. As $V \in \mathbb{R}^{m\times m}$ in (95), we may take an arbitrary symmetric positive semidefinite matrix satisfying the second condition in (11) for the matrix $A_{BC}^{+} - X_{k+1}$, and as $H \in \mathbb{R}^{n\times n}$ in (103), we may take an arbitrary symmetric positive semidefinite matrix satisfying the first condition in (11) for the matrix $A_{BC}^{+} - X_{k+1}$. It is easy to verify that by virtue of (39) and (94), the matrix $B_{EE}^{+1/2}$ satisfies the second condition in (11) for the matrix $A_{BC}^{+} - X_{k+1}$, where the matrices X_{k+1} are defined by (94), and the matrix C_{EE}^{+} by virtue of (39) and (102) satisfies the first condition in (11) for the matrix $A_{BC}^{+} - X_{k+1}$, where the matrices X_{k+1} are defined by (102). Then if we put $V = B_{EE}^{+1/2}$ in (95) and $H = C_{EE}^{+}$ in (103), then the formulas (95) and (103) are identical.

4. ITERATIVE METHODS FOR CALCULATING WEIGHTED NORMAL PSEUDOSOLUTIONS

First let us consider an iterative process for calculating weighted normal pseudosolutions with positive definite weights. Put $x_k = X_k f$, where the matrices X_k are defined by (81). Then we obtain the iterative process

$$x_0 = 0; \quad x_{k+1} = \Psi^{-p} x_k + \alpha \sum_{i=1}^{p} \Psi^{-i} C^{-1} A^T Bf,$$

$$\Psi = E + \alpha C^{-1} A^T BA, \quad k = 0, 1, \dots, \tag{108}$$

for the approximation to $x^{+} = A_{BC}^{+} f$.

THEOREM 8. The iterative process (108) for $0 < \alpha < \infty$ converges in $\mathbb{R}^n(C)$, and the estimate

$$||x^{+} - x_{k+1}||_C \le q^{p(k+1)} ||x^{+}||_C \tag{109}$$

holds, where q and the matrix C are defined in Theorem 4.

Proof. Taking into account the equalities $x^{+} = A_{BC}^{+} f$, $x_k = X_k f$, and (85), we obtain

$$x^{+} - x_{k+1} = (A_{BC}^{+} - X_{k+1}) f = (A_{BC}^{+} A\Psi^{-1})^{p} (x^{+} - x_k),$$

$$\Psi = E + \alpha C^{-1} A^T BA, \quad k = 0, 1, \dots,$$

whence based on (20) and Remark 2 we have

$$||x^{+} - x_{k+1}||_C \le ||(A_{BC}^{+} A\Psi^{-1})^{p}||_C \, ||x^{+} - x_k||_C, \quad k = 0, 1, \dots. \tag{110}$$

In [13] (Lemma 4) it is shown that the norm of a matrix to the nth power, left-symmetrizable with the positive definite symmetrizer C, is equal to the norm of this matrix weighted to the power n with the weight C. Since the matrix $(A_{BC}^{+} A\Psi^{-1})^{p}$ is left-symmetrizable with the symmetrizer C (see proof of Theorem 4), by virtue of the aforesaid from (110) we obtain

$$||x^{+} - x_{k+1}||_C \le ||A_{BC}^{+} A\Psi^{-1}||_C^{p} \, ||x^{+} - x_k||_C, \quad k = 0, 1, \dots. \tag{111}$$

The spectral radius of the matrix $A_{BC}^{+} A\Psi^{-1}$ is defined by (83). In [2] it was established that for the matrix L symmetrizable with the symmetrizer C we have $||L||_{CC^{-1/2}} \equiv ||L||_C = \rho(L)$. By virtue of these circumstances, taking into account the initial approximation of the iterative process (108), from (111) we obtain (109), as was to be shown.

Now let us consider an iterative process for calculating weighted normal pseudosolutions with degenerate weights. Put $x_k = X_k f$, where the matrices X_k are defined by (94). Then we obtain the iterative process

$$x_0 = 0; \quad x_{k+1} = \Psi^{-p} x_k + \alpha \sum_{i=1}^{p} \Psi^{-i} CA^T Bf,$$

$$\Psi = E + \alpha CA^T BA, \quad k = 0, 1, \dots \tag{112}$$

for calculating the approximation to $x^{+} = A_{BC}^{+} f$.

THEOREM 9. The iterative process (112) for $0 < \alpha < \infty$ converges in $\overline{\mathbb{R}}^n (C_{EE}^{+})$, and the estimate

$$||x^{+} - x_{k+1}||_{C_{EE}^{+}} \le q^{p(k+1)} ||x^{+}||_{C_{EE}^{+}} \tag{113}$$

holds, where q and the matrix C are defined in Theorem 6.

Proof. Taking into account the equalities $x^{+} = A_{BC}^{+} f$, $x_k = X_k f$, and (98), we obtain

$$x^{+} - x_{k+1} = (A_{BC}^{+} - X_{k+1}) f = (A_{BC}^{+} A\Psi^{-1})^p (x^{+} - x_k),$$

$$\Psi = E + \alpha CA^T BA, \quad k = 0, 1, \dots . \tag{114}$$

We will estimate the error $z = x^{+} - x_{k+1}$ in the norm $||\cdot||_{C_{EE}^{+}}$. To use relation (20) for the estimate $(A_{BC}^{+} A\Psi^{-1})^p (x^{+} - x_k)$, it is necessary to show that conditions (15) with $H = C_{EE}^{+}$ hold for the matrix $L = (A_{BC}^{+} A\Psi^{-1})^p = A_{BC}^{+} A\Psi^{-p}$, i.e.,

$$rk(C_{EE}^{+} L) = rk(L), \quad rk(LC_{EE}^{+}) = rk(L), \tag{115}$$

and $x^{+} \in \overline{\mathbb{R}}^n (C_{EE}^{+})$.

Let us show equalities (115). To this end, we use the fact that conditions (32) are fulfilled for the matrix L in the proof of Theorem 6. Then by virtue of the second condition in (32) we have $rk(L) = rk(LC) = rk(LCC_{EE}^{+} C)$ $\le rk(LCC_{EE}^{+}) \le rk(LC_{EE}^{+}) \le rk(L)$, i.e., $rk(LCC_{EE}^{+}) = rk(LC_{EE}^{+}) = rk(L)$ and the second condition in (115) is fulfilled. Now based on the equality $rk(LCC_{EE}^{+}) = rk(L)$ and the first condition in (115) we obtain $rk(L) = rk(LCC_{EE}^{+})$ $= rk(CL^T C_{EE}^{+}) \le rk(L^T C_{EE}^{+}) = rk(C_{EE}^{+} L) = rk(L)$, i.e., $rk(C_{EE}^{+} L) = rk(L)$, thus the first condition in (115) is fulfilled too.

The condition for the pseudosolution $x^{+} \in \overline{\mathbb{R}}^n (C_{EE}^{+})$ follows from its representation $x^{+} = A_{BC}^{+} f$, formula (39), and the obvious equality $C^{1/2} C_{EE}^{+1/2} C = C$. Then by virtue of (20) for the vector $z = x^{+} - x_k$ defined by (114), we obtain

$$||x^{+} - x_k||_{C_{EE}^{+}} \le ||(A_{BC}^{+} A\Psi^{-1})^p||_{C_{EE}^{+}} \, ||x^{+} - x_k||_{C_{EE}^{+}}. \tag{116}$$

In the proof of Theorem 6 it was mentioned that Lemma 4 holds for the matrix $(A_{BC}^{+} A\Psi^{-1})^p$, based on which from (116) we have

$$||x^{+} - x_k||_{C_{EE}^{+}} \le ||A_{BC}^{+} A\Psi^{-1}||_{C_{EE}^{+}}^p \, ||x^{+} - x_k||_{C_{EE}^{+}}. \tag{117}$$

Since $||A_{BC}^{+} A\Psi^{-1}||_{C_{EE}^{+}} \equiv ||A_{BC}^{+} A\Psi^{-1}||_{C_{EE}^{+} C^{1/2}} = \rho(A_{BC}^{+} A\Psi^{-1})$ (see [3]), with the initial approximation of the iterative process and (99), from (117) we obtain estimate (113), as was to be shown.

5. ITERATIVE METHODS FOR SOLVING CONSTRAINED LEAST-SQUARES PROBLEMS

In a number of studies (see, for example, [23, 24]), the solution of some constrained least-squares problems, L-pseudosolution [25], and Lg-pseudosolution [26] are presented using ML-weighted pseudoinverse matrices.

Let us define ML-weighted pseudoinverse matrices. Let A, M, and L be $m\times n$, $q\times m$, and $p\times n$ matrices, respectively. Then the ML-weighted pseudoinverse matrix A^{+}_{ML} for the matrix A is defined by the relation [23, 24, 27]

$$A^{+}_{ML} = (E-(LP)^{+}_{EE}L)(MA)^{+}_{EE}M, \quad P = E-(MA)^{+}_{EE}MA. \tag{118}$$

Let us define a weighted pseudoinverse matrix for the matrix A with the positive semidefinite weights B and C^{+}_{EE} as a matrix satisfying the system of matrix equations

$$AXA = A, \quad XAX = X, \quad (BAX)^{T} = BAX, \quad (XAC^{+}_{EE})^{T} = XAC^{+}_{EE} \tag{119}$$

provided that

$$rk\,(BA) = rk\,(A), \quad rk\,(AC^{+}_{EE}) = rk\,(A). \tag{120}$$

In [3] it was established that when the conditions

$$B = M^{T}M, \quad C^{+}_{EE} = (L^{T}L)^{+}_{EE}, \quad rk\,(M^{T}MA) = rk\,(A), \quad rk\,(A\,(L^{T}L)^{+}_{EE}) = rk\,(A),$$

$$(Bu,u)_{E_m} \geq 0, \quad (C^{+}_{EE}\,\upsilon,\upsilon)_{E_n} \geq 0, \quad \forall\, u \neq 0 \in \mathbb{R}^{m}, \quad \upsilon \neq 0 \in \mathbb{R}^{n}, \tag{121}$$

hold, the ML-weighted pseudoinverse matrix (118) is a weighted pseudoinverse matrix defined by relations (119) and (120).

Our purpose is to develop iterative methods for solving constrained least-squares problem and problems of calculating L-pseudosolution (Lg-pseudosolution) using the iterative methods (developed and analyzed in Sec. 4) for calculating weighted normal pseudosolutions and condition (121), for which ML-weighted pseudoinverse matrices coincide with weighted pseudoinverse matrices with degenerate weights. Hereafter we will assume that these conditions are satisfied.

First let us consider the least-squares problem with the following linear equalities as constraints [23, 24, 27]:

$$\min_{f\in\Omega} ||Kf-g||_{E}, \quad \Omega = \{f\,|\,Lf = h\}. \tag{122}$$

Conditions whereby problem (122) has a unique solution were established in [23]. We will assume that these conditions are fulfilled and that the matrix $K^{T}K$ is degenerate and the condition

$$rk\,(A\,(K^{T}\,K)^{+}_{EE}) = rk\,(A) \tag{123}$$

holds for it.

Then by virtue of (121) the solution of problem (122) is determined by the formula [23]

$$f_{*} = L^{+}_{EC^{+}_{EE}}h + (K\,P_{L})^{+}_{EE}g, \quad C = K^{T}K, \quad P_{L} = E - L^{+}_{EE}L. \tag{124}$$

Thus, when condition (123) is satisfied, the solution of problem (122) is the sum $f_{*} = f^{(1)}_{*} + f^{(2)}_{*}$ of normal pseudosolutions of two problems: (i) find the weighted normal pseudosolution of the system $Lf^{(1)} = h$ with the positive definite weight E and degenerate weight $C^{+}_{EE} = (K^{T}\,K)^{+}_{EE}$ (or find the ML-weighted normal pseudosolution of this system with $M = E$ and $L = K$) and (ii) find the normal pseudosolution of the system $KP_{L}\,f^{(2)} = g$. Then based on the iterative process (112), we have the iterative process

$$f_0^{(1)} = 0; \quad f_{k+1}^{(1)} = \Psi^{-p} f_k^{(1)} + \alpha \sum_{i=1}^{p} \Psi^{-i} C_{EE}^{+} L^T h,$$

$$\Psi = E + \alpha C_{EE}^{+} L^T L, \quad 0 < \alpha < \infty, \quad k = 0, 1, \ldots, \tag{125}$$

for the approximate calculation of $f_*^{(1)}$, and the iterative process

$$f_0^{(2)} = 0; \quad f_{k+1}^{(2)} = \Psi^{-p} f_k^{(2)} + \alpha \sum_{i=1}^{p} \Psi^{-i} (K P_L)^T g,$$

$$\Psi = E + \alpha (KP_L)^T KP_L, \quad 0 < \alpha < \infty, \quad k = 0, 1, \ldots \tag{126}$$

for the calculation of $f_*^{(2)}$.

Note that a weighting method of is proposed in [27] and regularized problems for the solution of problem (122) in [12, 15]. Computer-algebraic procedures for the solution of this problem are developed in [28].

Let us consider the least-squares problem with the following quadratic inequalities as constraints [23]:

$$\min_{f \in \Omega} \|Kf - g\|_E, \quad \Omega = \{f \mid \|f\|_N \le \omega\}, \quad N = L^T L, \tag{127}$$

and a special case of this problem:

$$\min_{x \in \Omega^*} \|Ax - b\|_E, \quad \Omega^* = \{x \mid \|x\|_E = \omega\}. \tag{128}$$

In [23] conditions are defined for which problem (127) has a unique solution, determined using *ML*-weighted pseudoinverse matrices. Provided that conditions (121) and (123) are satisfied, we obtain the following formula for calculating this solution:

$$f_* = L_{EC_{EE}^{+}}^{+} x_* + (KP_L)_{EE}^{+} g, \quad C = K^T K, \quad P_L = E - L_{EE}^{+} L, \tag{129}$$

where x is the solution of problem (128) for

$$A = KL_{EC}^{+}, \quad P_L = E - L_{EE}^{+} L, \quad Q_N = E - KP_L (KP_L)_{EE}^{+}, \quad b = Q_N g. \tag{130}$$

Efficient methods (see, for example, [29, 30]) are developed for the solution of problem (128). Then if problem (128) with (130) is solved, then the solution of problem (127) can be represented according to (129) by the sum $f_* = f_*^{(1)} + f_*^{(2)}$ of the weighted normal pseudosolution of the problem $Lf^{(1)} = x_*$ with the weights E and $C_{EE}^{+} = (K^T K)_{EE}^{+}$ and the normal pseudosolution of the problem $KP_L f^{(2)} = g$. Therefore, it is possible to use (accurate to the notation) iterative processes (125) and (126) for the approximate calculation of f_*.

The L-pseudosolution of the equation $Au = f$ as the solution of the problem

$$\min_{u \in \Omega} \|Lu - g\|_E, \quad \Omega = \{u \mid \min_{u \in \mathbb{R}^n} \|Au - f\|_E\} \tag{131}$$

and the conditions under which the problem (131) has a unique solution are defined in [25].

In [26], Lg-pseudosolution of the system $Au = f$ is defined as a unique solution of the problem

$$u_* = \underset{u \in \Omega_1 \subset \Omega_2}{\arg\min} \|u\|_E, \quad \Omega_1 = \{u \mid \min_{u \in \Omega_2} \|Lu - g\|_E\},$$

$$\Omega_2 = \{u \mid \min_{u \in \mathbb{R}^n} \|Au - f\|_E\}. \tag{132}$$

In [31], the solution x_0 of the problem

$$x_0 \in \operatorname{Arg} \min_{X} \|Ax - y\|_E^2, \quad X = \operatorname{Arg} \min_{x \in \mathbb{R}^n} \|Bx - z\|_E^2 \tag{133}$$

with the minimum norm is called a coupled normal pseudosolution.

We will assume that the matrices $L^T L$ ($A^T A$ for the problem (133)) are degenerate, and the following conditions are satisfied for them:

$$rk(A(L^T L)^+_{EE}) = rk(A), \quad (rk(B(A^T A)^+_{EE}) = rk(B) \text{ for problem (133)}). \tag{134}$$

Then provided that conditions (134) hold, from [24] we get the representation of the *L*-pseudosolution with the minimum norm and *Lg*-pseudosolution of the system $Au = f$

$$u_* = A^+_{EN^+_{EE}} f + (LP_A)^+_{EE} g, \quad N = L^T L, \quad P_A = E - A^+_{EE} A, \tag{135}$$

and the representation

$$x_* = B^+_{EN^+_{EE}} z + (AP_B)^+_{EE} y, \quad N = A^T A, \quad P_B = E - B^+_{EE} B \tag{136}$$

for the coupled normal pseudosolution of the problem (133).

From (135) and (136) it follows that *L*- and *Lg*-pseudosolutions and a coupled normal pseudosolution are also determined by the sum of weighted normal pseudosolution and the orinary normal pseudosolution; it is possible to use (accurate to notation) iterative processes (125) and (126) for their approximate solution.

Note that regularized problems are proposed and analyzed in [12, 15, 25] for finding an approximate solution of problem (131).

In [31], a two-parameter regularized problem is proposed for approximating the solution of problem (133) with a minimum norm.

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